1. Solve problem 6.13 in Dobrow. For each part, find $E[N_s|N_t]$.
   
   *Hint.* For the first part, look at the solution to 6.1 from last time. Show that the property illustrated there works for general $s < t$. See also 1.23 from HW #2.

2. Consider a post office with two clerks. Three people, A, B and C, enter at the same time. A and B go directly to the clerks, so C must wait until one of them leaves before beginning service. In each of the following scenarios, find the chance that A is still in the post office when the other two have completed service.

   (a) The service times are exactly (nonrandom) ten minutes.
   (b) The service times are independent and last $i$ minutes with probability $1/3$, for $i = 1, 2, 3$.
   (c) The service times have independent exponential distributions, with parameters $\alpha, \beta, \gamma$ for customers A, B and C, respectively. (*Hint.* Think about our discussion of exponential “alarm clocks”.)

3. Complete the steps below to fill in details for the proof that Defn II $\Rightarrow$ Defn I for the Poisson Process. Recall that in Defn II we have $S_n = X_1 + \cdots + X_n$, where the $X_i$ are i.i.d. $\text{Expo}(\lambda)$ random variables. Then $S_n$ has the Gamma($n, \lambda$) distribution density

   $$f(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \quad t > 0$$

   The following steps will show that $N_t \sim \text{Pois}(\lambda t)$.

   (a) Recall from class that for any counting process we have count-time duality:

   $$N_t \geq n \iff S_n \leq t.$$ 

   Use this to explain why $P(N_t = n) = P(S_n \leq t) - P(S_{n+1} \leq t)$.

   (b) Express the terms on the right in (b) as integrals of appropriate gamma densities.

   (c) Finally, perform an integration by parts on the second integral. The required formula for $P(N_t = n)$ should now emerge.

4. In Definition III for the Poisson process, we derived the following system of differential equations for $P_k(t) = P(N_t = k)$:

   $$\frac{d}{dt}P_0(t) = -\lambda P_0(t);$$

   $$\frac{d}{dt}P_k(t) = -\lambda P_k(t) + \lambda P_{k-1}(t), \quad k = 1, 2, \ldots.$$
We solved these equations by induction. For another approach, define the generating function for $N_t$ by setting
\[
G(s, t) = G_{N_t}(s) = E[s^{N_t}] = \sum_{k=0}^{\infty} s^k P_k(t)
\]
(More precisely, this is a family of generating functions, one for each value of $t$.)

(a) Take the term-by-term derivative with respect to $t$ of the sum in the definition of $G(s, t)$. This is actually a partial derivative $\partial / \partial t$, treating $s$ as a constant.

Then substitute from the $\frac{d}{dt} P_k(t)$ equations, $k = 0, 1, \ldots$, to show that
\[
\frac{\partial G}{\partial t} = \lambda(s - 1)G
\]

(b) Explain why the equation in (a) has solution $G(s, t) = \exp\{\lambda t(s - 1)\}$.

(c) Why can you now conclude that $N_t \sim \text{Pois}(\lambda t)$?