Read: Dobrow, Chapter 7, §5 (starting p.297 “Birth-and-Death”) and §6; Chapter 8, §1.

Problems: p. 316, # 7.19, 7.20; p. 366, # 8.1, 8.27.

1. Polya’s Urn revisited. Let $T$ be the random time at which the first red ball is drawn. At the completion of this step there will be 2 red balls in the urn, so $T = \min\{n : X_n = 2\}$.

Refer to problem 5 on HW#6. You showed that $P(T > n) = \frac{1}{n+1} \to 0$, but $E[T] = +\infty$.

(a) Show that the PMF of $T$ is given by

$$P(T = n) = \frac{1}{n(n+1)}, \quad n = 1, 2, \ldots$$

(b) It follows that

$$E[1/(T + 2)] = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}.$$ 

Why?

(c) Without evaluating this sum, use the Series Comparison Test from Calculus II to prove that it is finite.

(d) Now find the value of the sum as follows. Recall that setting $M_n = X_n/(n + 2)$ defines a martingale $\{M_n, n \geq 0\}$ with respect to $\{X_n, n \geq 0\}$. Use the Martingale Stopping Theorem (first making sure that it applies!) to show that $E[1/(T + 2)] = 1/4$.

2. Recall our set-up for the Gambler’s Ruin: $S_0 = k$ and $S_n = S_0 + X_1 + \cdots + X_n$ where the $X_i$ are independently $\pm 1$ with respective probabilities $p$ and $q$. Let $T = \min\{n : S_n = 0 \text{ or } S_n = N\}$.

(a) Show that $M_n = S_n - n(p - q)$ defines a martingale $\{M_n\}$ with respect to $\{X_n\}$.

(b) Apply the Martingale Stopping Theorem to show that

$$E[T] = \frac{k}{q - p} - \frac{N}{q - p} \left[ \frac{1 - (q/p)^k}{1 - (q/p)^N} \right], \quad 0 \leq j \leq N.$$ 

What conditions ensure that the theorem applies?

3. Consider the two period stock model from class, with zero interest rate.

(a) Find the no-arbitrage price at $t = 0$ for a European call option with strike price $K = 110$ and expiry $T = 2$.

(b) Same question for $K = 90$ and $T = 2$. 