Math 223: Multivariable Calculus
Example of the Second Derivative Test

Sometimes the algebra of finding critical points can be a little messy. Here is a simple example that highlights a useful trick to keep in mind.

**Example 0.1.** Suppose \( f(x, y) = x^2y - 4xy + y^2 \). Find the critical points of \( f(x, y) \).

We must find all points \( \bar{a} \) such that \( Df(\bar{a}) = \begin{bmatrix} 0 & 0 \end{bmatrix} \). This amounts to finding all points \( \bar{a} \) such that \( f_x(\bar{a}) = f_y(\bar{a}) = 0 \). We compute

\[
\begin{align*}
  f_x(x, y) &= 2xy - 4y = 2y(x - 2) \\
  f_y(x, y) &= x^2 - 4x + 2y
\end{align*}
\]

Here is a useful trick/technique for solving a system like this: Notice that \( f_x \) in this case can be written as a simple product and observe that the only way that the product can equal 0 is if one or the other of the factors equals 0.

Thus \( f_x(x, y) = 0 \) when \( y = 0 \) or when \( x = 2 \). This allows us to break into two cases.

Case 1: \( y = 0 \). In this case, \( f_y(x, y) = x^2 - 4x + 2(0) = x^2 - 4x = x(x - 4) \). So \( f_y(x, y) \) will also equal 0 if \( x = 0 \) or \( x = 4 \). Thus we have two critical points: \((0, 0)\) and \((4, 0)\).

Case 2: \( x = 2 \). In this case, \( f_y(x, y) = (2)^2 - 4(2) + 2y = -4 + 2y \). So \( f_y(x, y) \) will also equal 0 if \( y = 2 \). Thus we have a third critical point: \((2, 2)\).

Now that we have the critical points, we can classify each one. We’ll just do one here:

**Example 0.2.** Classify the critical point \((2, 2)\) of \( f(x, y) = x^2y - 4xy + y^2 \).

The Hessian of \( f \) is given by

\[
Hf(x, y) = \begin{bmatrix} 2y & 2x - 4 \\ 2x - 4 & 2 \end{bmatrix}.
\]

At \( \bar{a} = (2, 2) \), this is

\[
Hf(2, 2) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.
\]

Thus \( \det Hf(2, 2) = 8 > 0 \). Furthermore, \( f_{xx}(2, 2) = 4 > 0 \). Therefore, the second derivative test tells us that \( f \) has a local minimum at \((2, 2)\).